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Distributional Properties of the Three-Dimensional Poisson Delaunay Cell

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This paper gives distributional properties of geometrical characteristics of the Delaunay tessellation generated by a stationary Poisson point process in $\mathbb{R}^3$. The considerations are based on a well-known formula given by Miles which describes the size and shape of the "typical" three-dimensional Poisson Delaunay cell. The results are the probability density functions for its volume, the area, and the perimeter of one of its faces, the angle spanned in a face by two of its edges, and the length of an edge. These probability density functions are given in integral form. Formulas for higher moments of these characteristics are given explicitly.

KEY WORDS: Delaunay tessellation; Poisson Delaunay cell; Poisson point process; probability density functions; moments.

1. INTRODUCTION

The Delaunay tessellation is an important model for the approximation of real structures in a wide field of research. It is a space-filling subdivision of the d-dimensional Euclidean space $\mathbb{R}^d$ into d-dimensional simplices, whose vertices are the points of a point process. In general, any $(d+1)$-tuple of points of a point process generates a d-dimensional ball. A cell of the Delaunay tessellation is generated by such a $(d+1)$-tuple, if and only if the d-dimensional ball does not contain another point of the point process. The Delaunay cells are triangles in the two-dimensional ($d=2$) and tetrahedrons in the three-dimensional ($d=3$) case. These cases are important for applications of this model in crystallographic studies, for approximations in the continuum and quantum field theory, for studies of the

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mechanical response of heterogeneous materials, and in many other fields. Much of this work is summarized in Okabe *et al.*,[1] Kumar and Kurtz,[2] and references therein. The results are obtained mainly by simulation.

Under the assumption that the generating point process of the Delaunay tessellation is a Poisson point process, a theoretical investigation is possible by methods of integral geometry[3] or by the theory of Palm measures.[4] The distributional properties of the size and shape of a Poisson Delaunay cell for an arbitrary dimension are completely described by Miles,[3] formula (76). This result is used as a fundamental relation for the determination of the geometrical characteristics for the three-dimensional Deleunay tessellation generated by a stationary Poisson process.

Formula (76) in Miles[3] (in the following called Miles' formula) has been often used for the two-dimensional case (e.g., refs. 5–8). In contrast, Miles' formula has rarely been used for the three-dimensional Poisson Delaunay cell, because the number of integrations increases up to nine. Simulation studies of geometrical characteristics are made in Kumar and Kurtz[2] for the three-dimensional case.

The present paper gives analytical results for geometrical characteristics of the three-dimensional Poisson Delaunay cell. The considerations are based on a modification of Miles' formula given by Muche,[9] which has a more suitable form for an analytical treatment.

Probability density functions are given for the following characteristics of the three-dimensional Poisson Delaunay cell:

- the cell volume and the equivalent radius (radius of a ball of equivalent volume)
- the area of a face
- the perimeter of a face
- the length of an edge
- an angle in a face spanned by two of its edges

In general, such probability density functions cannot be given explicitly, but in the form of multiple integrals. The graphs of these functions are obtained by numerical integration. Expressions are given for the moments of these characteristics explicitly. A short summary is given for the one- and two-dimensional cases as well.

2. MILES' FORMULA

Let $\mathfrak{D}$ denote the Delaunay tessellation with respect to a stationary Poisson point process Φ in $\mathbb{R}^d$ with intensity λ. Let D denote the "typical" cell of $\mathfrak{D}$. The work "typical" is used as in Stoyan *et al.*,[10] p. 110. This "typical" cell D of $\mathfrak{D}$ is equivalent to the set of all inner points of a d-dimensional simplex spanned by $d+1$ points z_1, $z_2, ..., z_d$ and $z_{d+1} \in \Phi$ with the property

$$\|z_1\| = \|z_2\| = \cdots = \|z_d\| = \|z_{d+1}\| = \Delta$$

This means that these $d+1$ points are placed on the boundary of a d-dimensional ball $b(o, \Delta)$ of radius Δ centered in the origin o and there are no points of Φ closer to o.

Let U_i denote the projection of the point z_i ($i = 1, 2, ..., d+1$) onto the unit sphere $\partial b(o, 1)$, where $z_i = U_i \Delta$, $0 \leqslant \Delta < \infty$. Thus the size and shape of the "typical" Poisson Delaunay cell are completely characterized by the radius Δ of the ball and the $d+1$ unit vectors corresponding to the vertices of D. The corresponding probability density function is given by Miles' formula [ref. 3, formula (76)]

$$f_{\Delta, U_1, U_2, ..., U_{d+1}}(\delta, u_1, u_2, ..., u_{d+1}) = \kappa_d \delta^{d^2-1} \exp(-\lambda \omega_d \delta^d)\, v_d(u_1, u_2, ..., u_{d+1})$$

Here, $\omega_d = \pi^{d/2}/\Gamma(d/2+1)$ denotes the volume of the unit d-ball, $v_d(u_1, u_2, ..., u_{d+1})$ is the (in general positive) d-dimensional Lebesgue measure of the simplex spanned by $d+1$ unit vectors u_1, $u_2, ..., u_{d+1}$, and κ_d is a coefficient depending on d. Let Θ be a random variable defined for D which is invariant under Euclidean motions (for example, the d-dimensional Lebesgue measure of D or the length of one of its edges). Then the distribution function $F_\Theta(\theta)$ is given by

$$F_\Theta(\theta) = \kappa_d \underbrace{\int_0^\infty \overbrace{\int_{\partial b(o,1)} \cdots \int_{\partial b(o,1)}}^{d \text{ integrations}}}_{\Theta < \theta} \delta^{d^2-1} \exp(-\lambda \omega_d \delta^d)$$
$$\times\, v_d(u_1, u_2, ..., u_{d+1})\, du_{d+1} \cdots du_2\, du_1\, d\delta \qquad (2.1)$$

For the three-dimensional case (2.1) has been simplified by Muche,[9] using a Cartesian coordinate system (ξ, η, ζ) in such a way that the unit vectors have the coordinates

$$u_1 = (\xi_1, -\eta_1, \zeta_1), \qquad u_2 = (\xi_1, \eta_1, \zeta_1)$$
$$u_3 = (\xi_1, \eta_3, \zeta_3), \qquad u_4 = (\xi_4, \eta_4, \zeta_4)$$

with $\xi_4 \leqslant \xi_1$, $\eta_1 \geqslant 0$, and $\zeta_1 \leqslant \zeta_3$. Let u_i' ($i = 1, 2, 3, 4$) be the projection of u_i onto the (η, ζ) plane. Let α_1 and α_2 be the angles $u_1' o u_2'$ and $u_2' o u_3'$; β the angle spanned by u_1, o, and the positive ξ axis; γ the angle spanned by the negative ζ axis, o, and u_4'; and, finally, $h=\xi_1-\xi_4$ the height of the tetrahedron. Then (2.1) takes the simpler form

$$F_\Theta(\theta)=\frac{140}{9}\lambda^3\underbrace{\int_0^\infty\int_0^\pi\int_0^{1+\cos\beta}\int_0^{2\pi}\int_0^{2\pi-\alpha_1}\int_0^{2\pi}}_{\Theta<\theta}\delta^8\exp\left(-\frac{4\pi\lambda}{3}\delta^3\right)$$
$$\times h\sin^5\beta\left(\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\sin\frac{\alpha_1+\alpha_2}{2}\right)^2 d\gamma\, d\alpha_2\, d\alpha_1\, dh\, d\beta\, d\delta \tag{2.2}$$

The structure of the integrand allows a separation of $F_\Theta(\theta)$ into factors, namely

$$F_\Theta(\theta)=\underbrace{\int_0^\infty}_{\Theta<\theta} f_\Delta(\delta)\,d\delta\underbrace{\int_0^{2\pi}\int_0^{1+\cos\beta}}_{\Theta<\theta} f_{B,H}(\beta,h)\,dh\,d\beta$$
$$\times\underbrace{\int_0^{2\pi}\int_0^{2\pi-\alpha_1}}_{\Theta<\theta} f_{A_1,A_2}(\alpha_1,\alpha_2)\,d\alpha_2\,d\alpha_1\underbrace{\int_0^{2\pi}}_{\Theta<\theta} f_\Gamma(\gamma)\,d\gamma \tag{2.3}$$

with[9]

$$f_\Delta(\delta)=\frac{32\pi^3\lambda^3}{9}\delta^8\exp\left(-\frac{4\pi\lambda}{3}\delta^3\right),\qquad 0\leqslant\delta<\infty$$

$$f_{B,H}(\beta,h)=\frac{105}{64}h\sin^5\beta,\qquad 0\leqslant h<1+\cos\beta,\quad 0\leqslant\beta<\pi$$

$$f_{A_1,A_2}(\alpha_1,\alpha_2)=\frac{16}{3\pi^2}\left(\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\sin\frac{\alpha_1+\alpha_2}{2}\right)^2$$
$$0\leqslant\alpha_2<2\pi-\alpha_1,\quad 0\leqslant\alpha_1<2\pi$$

and

$$f_\Gamma(\gamma)=\frac{1}{2\pi},\qquad 0\leqslant\gamma<2\pi$$

The behavior of size and shape of the three-dimensional Poisson Delaunay cell D is completely described by formulas (2.2) and (2.3).

3. RESULTS

This section summarizes integral expressions for probability density functions $f_\Theta(\theta)$, formulas for the kth-order moment, $k=0, 1, 2,...$, the variance $\text{var}\,\Theta = E(\Theta - E\Theta)^2$, the skewness $\text{skw}\,\Theta = E(\Theta - E\Theta)^3/(\text{var}\,\Theta)^{3/2}$, and the excess $\text{exc}\,\Theta = E(\Theta - E\Theta)^4/(\text{var}\,\Theta)^2 - 3$ for several geometrical characteristics Θ of the three-dimensional Poisson Delaunay cell. The abbreviation

$$g(\alpha_1, \alpha_2) = \sin(\alpha_1/2)\sin(\alpha_2/2)\sin[(\alpha_1+\alpha_2)/2]$$

is used in the probability density functions.

Numerical values are summarized in Table I.

The probability density functions are plotted in Figs. 1–4.

Table I. Properties of Geometrical Characteristics of the Three-Dimensional Delaunay Cell to a Generating Poisson Process of Unit Intensity[a]

Θ	$E\Theta$	$E\Theta^2$	$E\Theta^3$	$E\Theta^4$
V	0.1477600595	0.0371983367	0.0134741017	0.0064177669
R	0.3039813467	0.1001274593	0.0352751158	0.0131649871
S	0.5972864450	0.4675444061	0.4444664589	0.4913927123
P	3.7111010836	14.6399982926	60.8389538054	264.5488393183
L	1.2370336945	1.7155937900	2.5850244273	4.1559502764
A	1.0471975512	1.2699340668	1.7136395601	2.5115877519
Θ	var Θ	sd Θ	skw Θ	exc Θ
V	0.0153653015	0.1239568533	1.8045024895	5.0345713432
R	0.0077228001	0.0878794636	0.2107951308	−0.1382798960
S	0.1107933087	0.3328562883	0.8909815462	0.9482501001
P	0.8677270400	0.9315186740	0.0841202890	−0.1304754550
L	0.1853414286	0.4305129831	0.0530118838	−0.3324898903
A	0.1733113556	0.4163068047	0.2880812204	−0.2833923139

[a] The parameter Θ stands for the cell volume V, the equivalent radius R of V, the area of a face S, the perimeter of a face P, the length of an edge L, and the angle inside of a face A (sd = standard deviation).

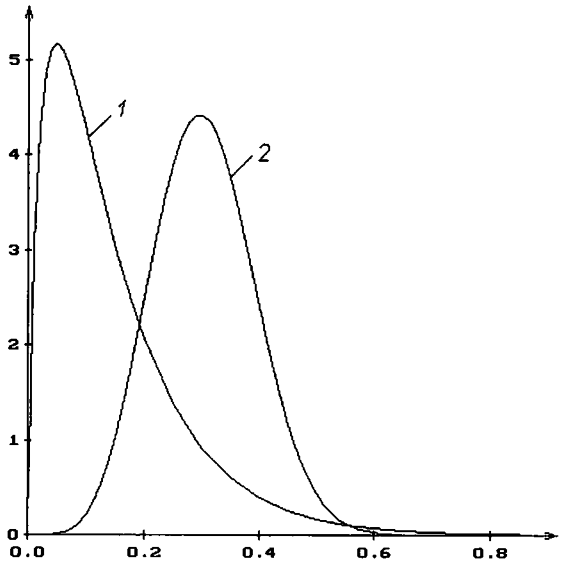

Fig. 1. The probability density function of the volume of the three-dimensional Poisson Delaunay cell (1) and that of the equivalent radius of volume (2). The intensity of the generating Poisson process is 1.

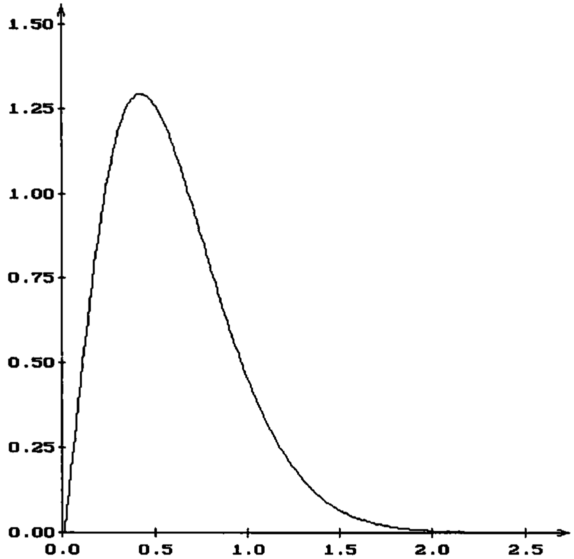

Fig. 2. The probability density function of the area of a face of the three-dimensional Poisson Delaunay cell. The intensity of the generating Poisson process is 1.

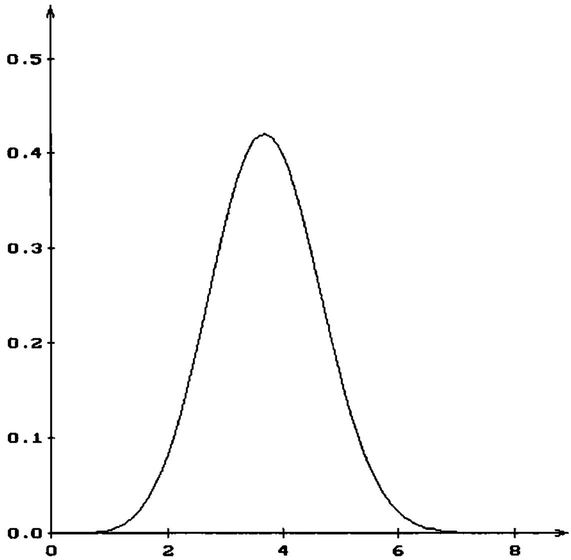

Fig. 3. The probability density function of the perimeter of a face of the three-dimensional Poisson Delaunay cell. The intensity of the generating Poisson process is 1.

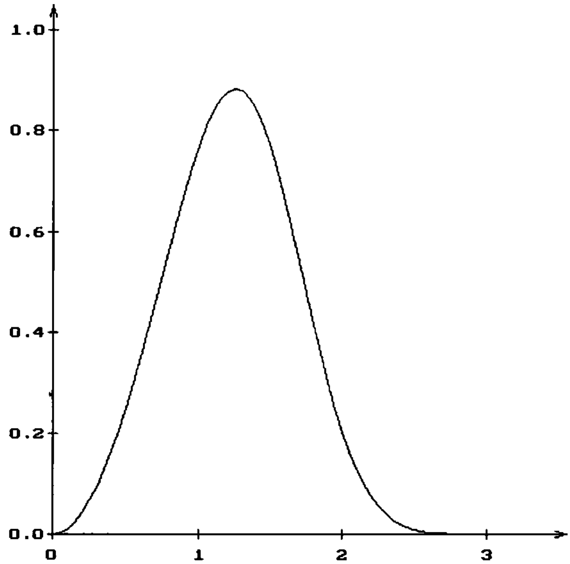

Fig. 4. The probability density functions of the length of an edge of the three-dimensional Poisson Delaunay cell. The intensity of the generating Poisson process is 1.

3.1. The Volume V

The results given in this section are based on formulas (2.2) and (2.3). First consider the volume of the Poisson Delaunay cel D of $\mathfrak{D}$ in $\mathbb{R}^3$. It is given by $\frac{1}{3}\cdot$ *area of lower surface* $\cdot$ *height*, which means substituting the volume $\Theta = V$ and $v = \frac{2}{3}\delta^3 g(\alpha_1, \alpha_2) h \sin^2\beta$ in (2.3). Transposition to δ gives

$$\delta = \left(\frac{3v}{2g(\alpha_1, \alpha_2) h \sin^2\beta}\right)^{1/3}$$

with the derivative

$$\frac{\partial\delta}{\partial v} = \frac{1}{3}\left(\frac{3}{2g(\alpha_1, \alpha_2) h \sin^2\beta \cdot v^2}\right)^{1/3}$$

Simplification and differentiation of (2.3) with respect to v gives

$$f_V(v) = 35\pi\lambda^3 \int_0^{2\pi}\int_0^{2\pi-\alpha_1}\int_0^{\pi}\int_0^{1+\cos\beta} \frac{v^2}{g(\alpha_1, \alpha_2)\, h^2 \sin\beta} \times \exp\left(-\frac{2\pi\lambda v}{g(\alpha_1, \alpha_2) h \sin^2\beta}\right) dh\, d\beta\, d\alpha_2\, d\alpha_1$$

The integration with respect to h by use of the substitution $h = 1/t$ leads to the probability density function of the volume of the three-dimensional Delaunay cell D

$$f_V(v) = \frac{35\lambda^2}{2}\int_0^{2\pi}\int_0^{2\pi-\alpha_1}\int_0^{\pi} v\sin\beta \times \exp\left(\frac{-2\pi\lambda v}{g(\alpha_1, \alpha_2)(1+\cos\beta)\sin^2\beta}\right) d\beta\, d\alpha_2\, d\alpha_1, \qquad v \geqslant 0 \quad (3.1)$$

Note that the slope of $f_V(v)$ at $v = 0$ is finite; the derivative takes the value $df_V(v)/dv = 70(\pi\lambda)^2$. The kth moment of V is well known [cf. Miles,[3] formula (77), for the special case $d = 3$, or Møller,[4] formula (7.35)]

$$EV^k = \frac{35\sqrt{\pi}(k+1)!\,(k+2)!\,(2k+4)!}{256\{\Gamma(k/2+2)\}^3\,\Gamma((3k+9)/2)(16\pi\lambda)^k} \qquad (3.2)$$

Variance, skewness, and excess are given by

$$\operatorname{var} V = \frac{30240\pi^2 - 175175}{82368\pi^4\lambda^2} \qquad \operatorname{skw} V = \frac{1}{20\sqrt{70}}\,\frac{(12250/864 - 96\pi^2/143)}{(3\pi^2/143 - 35/288)^{3/2}}$$

$$\operatorname{exc} V = \frac{6}{11305}\,\frac{2303976960\pi^4 + 323323(55296\pi^2 - 875875)}{(864\pi^2 - 5005)^2}$$

The equivalent radius R is connected with the volume by $V = \frac{4}{3}\pi R^3$. Therefore, the substitution $v = \frac{4}{3}\pi r^3$ in (3.1) leads to the probability density function of R,

$$f_R(r) = \frac{280}{3}(\pi\lambda)^2 r^5 \int_0^{2\pi} \int_0^{2\pi-\alpha_1} \int_0^{\pi} \sin\beta$$
$$\times \exp\left(\frac{-8\pi^2\lambda r^3}{3g(\alpha_1,\alpha_2)(1+\cos\beta)\sin^2\beta}\right) d\beta\, d\alpha_2\, d\alpha_1, \qquad r \geqslant 0$$

For the determination of the moments, the order of integrations can be changed. Now we use the well-known integral formulas

$$\int_0^{\infty} x^a \exp(-bx^c)\, dx = \frac{1}{c}\left(\frac{1}{b}\right)^{(a+1)/c} \Gamma\left(\frac{a+1}{c}\right), \qquad 0 < a, b, c < \infty \quad (3.3)$$

and

$$\int_0^{\pi/2} \sin^{2a+1} x \cos^{2b+1} x\, dx = \frac{\Gamma(a+1)\,\Gamma(b+1)}{2\Gamma(a+b+2)},$$
$$-\infty < a, b < +\infty \qquad (3.4)$$

This gives

$$EV^k = \frac{35}{8\pi^2}\frac{\Gamma(k+2)}{(2\pi\lambda)^k} \int_0^{2\pi} \int_0^{2\pi-\alpha_1} \int_0^{\pi} (1+\cos\beta)^{k+2} \sin^{2k+5}\beta$$
$$\times \{g(\alpha_1,\alpha_2)\}^{k+2}\, d\beta\, d\alpha_2\, d\alpha_1$$

and use of (3.2) leads to

$$\int_0^{2\pi} \int_0^{2\pi-\alpha_1} \{g(\alpha_1,\alpha_2)\}^c\, d\alpha_2\, d\alpha_1 = \frac{\pi^{5/2}\Gamma(3c+2)}{2^{6c}\Gamma((3c+3)/2)\{\Gamma(c/2+1)\}^3} \qquad (3.5)$$

for real $c \geqslant 0$. Use of (3.5) finally leads to a kth-moment formula for the equivalent radius

$$ER^k = \frac{35 \cdot 3^{k/3}\Gamma(k/3+2)\,\Gamma(k/3+3)\,\Gamma(2k/3+5)}{2^{2k+8}\pi^{2k/3-1/2}\{\Gamma(k/6+2)\}^3\,\Gamma((k+9)/2)\,\lambda^{k/3}}$$

with

$$\text{var } R = \frac{125}{(3\pi\lambda^2)^{1/3}}\left[\frac{91\{\Gamma(\frac{5}{3})\}^2}{17496\pi\{\Gamma(\frac{4}{3})\}^2} - \frac{605\pi^2\{\Gamma(\frac{4}{3})\}^2}{3^{14}\{\Gamma(\frac{7}{6})\}^6}\right]$$

$$\begin{aligned}\text{skw } R = {} & 5\{7\cdot 3^{43/2}\{\Gamma(\tfrac{4}{3})\}^3\{\Gamma(\tfrac{7}{6})\}^9 \\ & + 440000\pi^{9/2}[605\pi\{\Gamma(\tfrac{4}{3})\}^6 - 2457\{\Gamma(\tfrac{7}{6})\}^6]\} \\ & \times\{\sqrt{2}\,\pi[24877125\{\Gamma(\tfrac{5}{3})\}^2\{\Gamma(\tfrac{7}{6})\}^6 - 605000\pi^3\{\Gamma(\tfrac{4}{3})\}^4]^{3/2}\}^{-1}\end{aligned}$$

$$\begin{aligned}\text{exc } R = {} & \{[1233792\{\Gamma(\tfrac{4}{3})\}^6 - 2640625\{\Gamma(\tfrac{5}{3})\}^6] \\ & \times 3^{19}\cdot 49\{\Gamma(\tfrac{7}{6})\}^{12} - 8\cdot(11)^4(10\pi)^8\{\Gamma(\tfrac{4}{3})\}^6 \\ & + 6342336(10\pi)^7\{\Gamma(\tfrac{7}{6})\}^6 - 779625\cdot 18^7(3\pi)^{5/2}\{\Gamma(\tfrac{4}{3})\}^3\{\Gamma(\tfrac{7}{6})\}^9\} \\ & \times\{[1125\Gamma(\tfrac{5}{3})][3^7\cdot 91\{\Gamma(\tfrac{7}{6})\}^6\{\Gamma(\tfrac{5}{3})\}^2 - 4840\pi^3\{\Gamma(\tfrac{4}{3})\}^4]\}^{-2}\end{aligned}$$

3.2. The Area *S* of a Face

The behavior of the area of a face can be investigated by use of (2.3). Putting $\Theta = S$ and using the probability density function

$$f_B(\beta) = \int_0^{1+\cos\beta} f_{B,H}(\beta, h)\, dh = \frac{105}{128}\sin^5\beta\,(1+\cos\beta)^2, \qquad 0 \leqslant \beta < \pi$$

gives the distribution function of the face area in the form

$$F_S(s) = \underbrace{\iiiint}_{S<s} f_{A_1,A_2}(\alpha_1, \alpha_2) f_B(\beta) f_\Delta(\delta)\, d\alpha_2\, d\alpha_1\, d\beta\, d\delta$$

Because of $s = 2(\delta\sin\beta)^2 g(\alpha_1, \alpha_2)$, the substitution $\delta = [s/g(\alpha_1, \alpha_2)/2]^{1/2}/\sin\beta$, simplification, and differentiation with respect to s lead to the probability density function of the area of a face

$$\begin{aligned} f_S(s) = {} & \frac{35\pi\lambda^3}{72\sqrt{2}}\int_0^{2\pi}\int_0^{2\pi-\alpha_1}\int_0^{\pi}\frac{s^{7/2}(1+\cos\beta)^2}{\{g(\alpha_1,\alpha_2)\}^{5/2}\sin^4\beta} \\ & \times\exp\left(\frac{-\sqrt{2}\,\pi\lambda s^{3/2}}{3\{g(\alpha_1,\alpha_2)\}^{3/2}\sin^3\beta}\right) d\beta\, d\alpha_2\, d\alpha_1, \qquad s \geqslant 0\end{aligned}$$

The moments are obtained by use of (3.3)–(3.5)

$$ES^k = \frac{35 \cdot 3^{2k/3}(k+4)\,\Gamma(2k/3+3)\,\Gamma(k+3)\,\Gamma(3k/2+4)}{2^{10k/3+9}\pi^{2k/3-1/2}\{\Gamma(k/2+2)\}^3\,\Gamma(k+9/2)\,\lambda^{2k/3}}$$

$$\text{var } S = \frac{175}{2(6\pi^2\lambda^2)^{2/3}}\left[\frac{\Gamma(4/3)}{11} - \frac{4375\{\Gamma(5/3)\}^2}{5832\pi^2}\right]$$

$$\text{skw } S = \frac{2\sqrt{77}(22 \cdot 3^{12}\pi^2 - 5 \cdot 10^4 \cdot 3^{9/2}\pi^3 + 5^9 77\{\Gamma(5/3)\}^3)}{125(18^3\pi^2\Gamma(4/3) - 77 \cdot 5^4\{\Gamma(5/3)\}^2)^{3/2}}$$

$$\begin{aligned}\text{exc } S = \{&[(8\pi)^2 - 1365] \cdot 3^{11}(44\pi)^2\,\Gamma(\tfrac{5}{3}) \\ &+ [(12\pi)^3\sqrt{3} - 77 \cdot 5^4\{\Gamma(\tfrac{5}{3})\}^3] \\ &\times 6006 \cdot 5^6\Gamma(\tfrac{5}{3}) - 18^6 \cdot 975\pi^4\{\Gamma(\tfrac{4}{3})\}^2\} \\ &\times\{325[18^3\pi^2\Gamma(\tfrac{4}{3}) - 77 \cdot 5^4\{\Gamma(\tfrac{5}{3})\}^2]^2\}^{-1}\end{aligned}$$

3.3. The Perimeter *P* of a Face

Analogously, use of $\Theta = P$ and

$$\delta = p\Big/\left[2\sin\beta\left(\sin\frac{\alpha_1}{2} + \sin\frac{\alpha_2}{2} + \sin\frac{\alpha_1+\alpha_2}{2}\right)\right]$$

leads to the probability density function of the perimeter of a face

$$\begin{aligned}f_P(p) = \frac{35\pi\lambda^3}{2^7 \cdot 9}\int_0^{2\pi}\int_0^{2\pi-\alpha_1}\int_0^{\pi} &\frac{p^8\{g(\alpha_1,\alpha_2)\}^2(1+\cos\beta)^2}{\sin^4\beta(\sin(\alpha_1/2)+\sin(\alpha_2/2)+\sin[(\alpha_1+\alpha_2)/2])^9} \\ &\times\exp\left(\frac{-\lambda\pi p^3}{6\sin^3\beta(\sin(\alpha_1/2)+\sin(\alpha_2/2)+\sin[(\alpha_1+\alpha_2)/2])^3}\right) \\ &\times d\beta\, d\alpha_2\, d\alpha_1, \qquad p > 0\end{aligned}$$

Use of (3.3), (3.4), and a sequence of lengthy but elementary integrations with respect to α_1 and α_2 leads to the further expressions

$$EP^1 = \frac{1715}{512}\,\Gamma\left(\frac{4}{3}\right)\left(\frac{6}{\pi\lambda}\right)^{1/3}$$

$$EP^2 = \frac{3200}{81}\left(\frac{128}{75\pi^2} + \frac{3}{32}\right)\Gamma\left(\frac{5}{3}\right)\left(\frac{6}{\pi\lambda}\right)^{2/3}$$

$$EP^3 = \frac{12525975}{2^{16}\pi\lambda} \qquad EP^4 = \frac{71680}{891}\left(\frac{435}{1024} + \frac{8192}{735\pi^2}\right)\Gamma\left(\frac{4}{3}\right)\left(\frac{6}{\pi\lambda}\right)^{4/3}$$

$$\operatorname{var} P=\left\{\left(\frac{16384+900\pi^2}{243\pi^2}\right)\Gamma\left(\frac{5}{3}\right)-\left[\frac{1715}{512}\Gamma\left(\frac{4}{3}\right)\right]^2\right\}\left(\frac{6}{\pi\lambda}\right)^{2/3}$$

$$\begin{aligned}\operatorname{skw} P=\;&210\pi^2\{3645\sqrt{3}\,\pi[12215808+5\cdot 7^8\{\Gamma(\tfrac{4}{3})\}^3]\\&-2^{21}49[2^{12}+(15\pi)^2]\}\\&\times\{2^{20}[2^{12}+(15\pi)^2]\,\Gamma(\tfrac{5}{3})-175\cdot 21^5\pi^2\{\Gamma(\tfrac{4}{3})\}^2\}^{-3/2}\end{aligned}$$

$$\begin{aligned}\operatorname{exc} P=\;&\frac{6}{77}\{56^7\cdot 30^2\cdot 11\sqrt{3}\,\pi^3\Gamma(\tfrac{4}{3})[2^{12}+(15\pi)^2]\\&-2^{11}\cdot 945^2\cdot 11179622417\pi^4\Gamma(\tfrac{4}{3})\\&+2^{59}(9\pi)^2\,\Gamma(\tfrac{4}{3})-99\cdot 7^{13}[45\pi\Gamma(\tfrac{4}{3})]^4\\&-2^{39}\cdot 77[2^{12}+(15\pi)^2]^2\,\{\Gamma(\tfrac{5}{3})\}^2\}\\&\times\{2^{20}[2^{12}+(15\pi)^2]\,\Gamma(\tfrac{5}{3})-175\cdot 2^{15}\pi^2\{\Gamma(\tfrac{4}{3})\}^2\}^{-2}\end{aligned}$$

3.4. The Length L of an Edge

Consider the length L of an edge of D. For the determination of its probability density function consider the central angles A_1 and A_2 occurring in (2.3). The edge length and one of these angles are connected by $l=\delta\sin(\alpha_1/2)$. Therefore knowledge of the probability density function

$$f_{A_1}(\alpha_1)=\int_0^{2\pi-\alpha_1} f_{A_1,A_2}(\alpha,\alpha_2)\,d\alpha_2$$

is needed, namely

$$f_{A_1}(\alpha_1)=\frac{1}{3\pi^2}[(2+\cos\alpha_1)(2\pi-\alpha_1)+3\sin\alpha_1](1-\cos\alpha_1)$$

$$0\leqslant\alpha_1<2\pi \tag{3.6}$$

Now the distribution function $F_L(l)$ is given by

$$F_L(l)=\underbrace{\iiint}_{L<l} f_{A_1}(\alpha_1)f_B(\beta)f_\Delta(\delta)\,d\alpha_1\,d\beta\,d\delta$$

Substituting $\delta=l/[2\sin(\alpha_1/2)]$ and taking into account that

$$\int_0^{2\pi}(2\pi-\alpha_1)\sin^c\frac{\alpha_1}{2}\,d\alpha_1=\int_0^{2\pi}\alpha_1\sin^c\frac{\alpha_1}{2}\,d\alpha_1=\pi\int_0^{2\pi}\sin^c\frac{\alpha_1}{2}\,d\alpha_1$$

for real c leads to the probability density function of the length of an edge of D,

$$f_L(l)=\frac{35\pi^2\lambda^3}{9\cdot 2^7}\int_0^{\pi/2}\int_0^{\pi/2}\frac{l^8(2-\sin^2\beta)(3-2\sin^2\varphi)}{\sin^4\beta\sin^7\varphi}$$
$$\times\exp\left(-\frac{\pi\lambda l^3}{6\sin^3\beta\sin^3\varphi}\right)d\varphi\,d\beta,\qquad l\geqslant 0$$

The moments are obtained by use of (3.3) and (3.4),

$$EL^k=\frac{35}{32}\frac{(k+8)(k+6)}{(k+7)(k+5)(k+3)}\Gamma\left(3+\frac{k}{3}\right)\left(\frac{6}{\pi\lambda}\right)^{k/3}$$

$$\operatorname{var}L=\frac{25}{9}\left(\frac{4}{9}\Gamma\left(\frac{5}{3}\right)-\left(\frac{7}{8}\right)^6\left\{\Gamma\left(\frac{4}{3}\right)\right\}^2\right)\left(\frac{6}{\pi\lambda}\right)^{2/3}$$

$$\operatorname{skw}L=\frac{126}{125}\{2^{17}3^6 11+3\cdot 5^3 7^8\{\Gamma(\tfrac{4}{3})\}^3-2^{21}5^3 7^2\pi/3^{5/2}\}$$
$$\times[2^{20}\Gamma(\tfrac{5}{3})-9\cdot 7^6\{\Gamma(\tfrac{4}{3})\}^2]^{-3/2}$$

$$\operatorname{exc}L=[2^{20}\cdot 7\sqrt{3}\,\Gamma(\tfrac{4}{3})(1078\cdot 70^3\pi-183827583\sqrt{3})$$
$$-2^{40}4125\{\Gamma(\tfrac{5}{3})\}^2-668250\cdot 7^{12}\{\Gamma(\tfrac{4}{3})\}^4]$$
$$\times\{1375[2^{20}\Gamma(\tfrac{5}{3})-9\cdot 7^6\{\Gamma(\tfrac{4}{3})\}^2]^2\}^{-1}$$

3.5. The Angle *A* in a Face Spanned by Two of Its Edges

Finally, (3.6) and $A_1=2A$, *central angle* $=2\cdot$ *angle at circumference*, lead immediately to the probability density function of the angle spanned in a face by two of its edges. The moments are obtained by integration by parts and (3.4),

$$f_A(\alpha)=\frac{4}{3\pi^2}[2(\pi-\alpha)(2+\cos 2\alpha)+3\sin 2\alpha]\sin^2\alpha,\qquad 0\leqslant\alpha<\pi$$

with

$$EA^1=\frac{\pi}{3},\qquad EA^2=\frac{\pi^2}{6}-\frac{3}{8},\qquad \operatorname{var}A=\frac{\pi^2}{18}-\frac{3}{8}$$

These results were already given by Kumar and Kurtz.[2] The further expressions are

$$EA^3 = \frac{16\pi^4 - 60\pi^2 - 105}{160\pi}, \qquad EA^4 = \frac{32\pi^4 - 180\pi^2 - 135}{480}$$

$$\text{skw } A = \frac{\pi^3/135 - 21/32\pi}{(\pi^2/18 - 3/8)^{3/2}}, \qquad \text{exc } A = \frac{3}{5} \frac{1485 - 16\pi^4}{(4\pi^2 - 27)^2}$$

4. THE LOWER DIMENSIONAL CASES

The method given here can be used in the same manner for the one- and two-dimensional Delaunay tessellations.

4.1. The One-Dimensional Delaunay Cell

In the case $d = 1$, (2.1) takes the very easy form

$$F_\Theta(\theta) = 2\lambda \underbrace{\int_0^\infty}_{\Theta < \theta} \exp(-2\lambda\delta)\, d\delta$$

which means that one-dimensional Delaunay cells are simply segments with a random length (one-dimensional volume) V. Substituting $\Theta = V$ and $v = 2\delta$ gives the probability density function

$$f_V(v) = \lambda \exp(-\lambda v), \qquad v \geqslant 0$$

with

$$EV^k = \left(\frac{1}{\lambda}\right)^k \Gamma(k+1)$$

and var $V = 1/\lambda^2$, skw $V = 2$, and exc $V = 6$. These results are well known.

4.2. The Two-Dimensional Delaunay Cell

In the case $d = 2$ the cells of $\mathfrak{D}$ are planar triangles having the radius Δ of the circumcircle and the central angles A_1, A_2, and $2\pi - A_1 - A_2$ (see Table II). Then Miles' formula (2.1) can be written as

$$F_{\Theta}(\theta) = \frac{4\pi\lambda^2}{3} \underbrace{\int_0^{\infty} \int_0^{2\pi} \int_0^{2\pi - \alpha_1}}_{\Theta < \theta} \delta^3 \exp(-\lambda\pi\delta^2)$$

$$\times \sin\frac{\alpha_1}{2} \sin\frac{\alpha_2}{2} \sin\frac{\alpha_1 + \alpha_2}{2} \, d\alpha_2 \, d\alpha_1 \, d\delta \tag{4.1}$$

Let V denote the area (two-dimensional volume) of D of $\mathfrak{D}$. Using again

$$g(\alpha_1, \alpha_2) = \sin(\alpha_1/2) \sin(\alpha_2/2) \sin[(\alpha_1 + \alpha_2)/2]$$

the connection between the triangle area, the radius of the circumcircle, and the central angles is $v = 2\delta^2 g(\alpha_1, \alpha_2)$. Connected with (4.1), this gives immediately the probability density function of the area V of D of $\mathfrak{D}$ for $d = 2$ (see Fig. 5),

$$f_V(v) = \frac{\pi}{6} \lambda^2 v \int_0^{2\pi} \int_0^{2\pi - \alpha_1} \frac{1}{g(\alpha_1, \alpha_2)} \exp\left(\frac{-\lambda\pi v}{2g(\alpha_1, \alpha_2)}\right) d\alpha_2 \, d\alpha_1, \qquad v \geqslant 0$$

Use of (3.3) and (3.5) leads to

$$EV^k = \frac{\Gamma(k/2 + 1)\, \Gamma((3k + 5)/2)}{3 \cdot 2^k \pi^{k - 1/2} \{\Gamma((k + 3)/2)\}^2 \lambda^k}$$

$$\text{var } V = \frac{35 - 2\pi^2}{8\pi^2\lambda^2}$$

$$\text{skw } V = \frac{2^{1/2}\pi(4\pi^2 - 15)}{(35 - 2\pi^2)^{3/2}}$$

$$\text{exc } V = \frac{2331 + 120\pi^2 - 24\pi^4}{(35 - 2\pi^2)^2}$$

Substituting $v = \pi r^2$, we immediately obtain the expression for the equivalent radius R (a circle having an area equivalent to the area of D),

$$f_R(r) = \frac{\pi^3}{3} \lambda^2 r^3 \int_0^{2\pi} \int_0^{2\pi - \alpha_1} \frac{1}{g(\alpha_1, \alpha_2)} \exp\left(\frac{-\lambda\pi^2 r^2}{2g(\alpha_1, \alpha_2)}\right) d\alpha_2 \, d\alpha_1, \qquad r \geqslant 0$$

Table II. Properties of Geometrical Characteristics of the Two-Dimensional Delaunay Cell to a Generating Poisson Process of Unit Intensity[a]

Θ	$E\Theta$	$E\Theta^2$	$E\Theta^3$	$E\Theta^4$
V	0.5000000000	0.4432801784	0.5699316580	0.9633982722
R	0.3637754544	0.1591549431	0.0799998190	0.0449136724
P	3.3953054526	13.2629119243	57.9388014224	277.7416286004
L	1.1317684842	1.5915494309	2.5938223012	4.7283219033
A	1.0471975512	1.3599120891	2.0261201264	3.3284062697
Θ	var Θ	sd Θ	skw Θ	exc Θ
V	0.1932801784	0.4396364162	1.8242427745	5.0561445543
R	0.0268223618	0.1637753396	0.5892595020	0.2502552768
P	1.7348128077	1.3171229281	0.4931047489	0.1724363860
L	0.3106495291	0.5573594254	0.5162778297	0.0618276348
A	0.2632893779	0.5131173140	0.3744793156	−0.3812822265

[a] The parameter Θ stands for the cell area V, the equivalent radius R of V, the perimeter of a cell P, the length of an edge L, and the angle spanned by two edges of a cell A (sd = standard deviation).

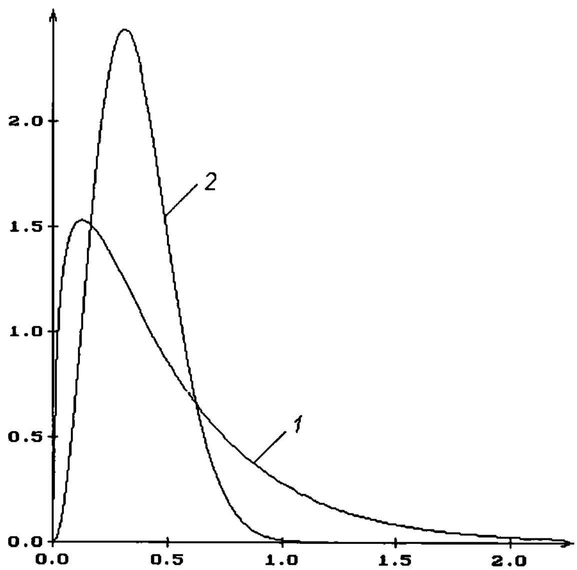

Fig. 5. The probability density function of the area of the two-dimensional Poisson Delaunay cell (1) and that of the equivalent radius of area (2). The intensity of the generating Poisson process is 1.

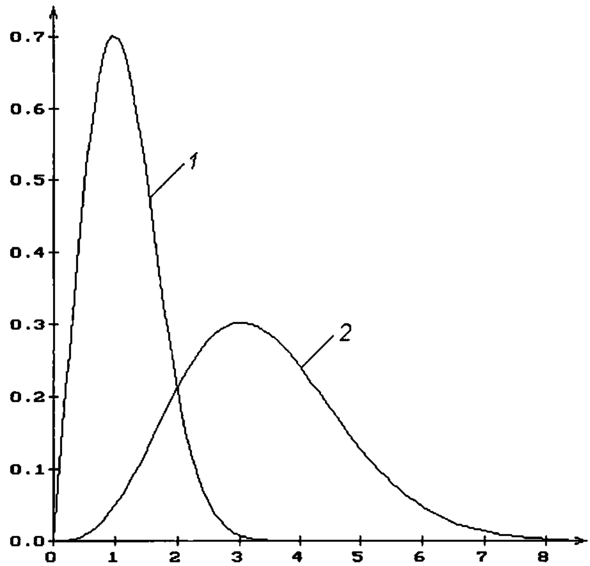

Fig. 6. The probability density function of the length of an edge (1) and the perimeter of a cell of the two-dimensional Poisson Delaunay tessellation. The intensity of the generating Poisson process is 1.

Use of (3.3) and (3.5) gives

$$ER^k = \frac{\Gamma((k+4)/4)\,\Gamma((3k+10)/4)}{3\cdot 2^{k/2}\pi^{k-1/2}\{\Gamma((k+6)/4)\}^2\,\lambda^{k/2}}$$

$$\text{var}\,R = \frac{1}{2\pi\lambda}\left[1-\left(\frac{40\{\Gamma(\frac{5}{4})\}^4}{3\pi^2}\right)^2\right]$$

$$\text{skw}\,R = \frac{2^{19}5^4\{\Gamma(\frac{5}{4})\}^{16} - 2^{12}675\pi^4\{\Gamma(\frac{5}{4})\}^8 + 3^5 77\pi^7}{2^9 5\{\Gamma(\frac{5}{4})\}^4(9\pi^4 - 1600\{\Gamma(\frac{5}{4})\}^8)^{3/2}}$$

$$\begin{aligned}\text{exc}\,R = {} & [3^4\pi^6(280 - 231\pi - 48\pi^2) \\ & + 2^{12}675\pi^4\{\Gamma(\tfrac{5}{4})\}^8 - 2^{17}3\cdot 5^4\{\Gamma(\tfrac{5}{4})\}^{16}] \\ & \times \{2^4[9\pi^4 - 1600\{\Gamma(\tfrac{5}{4})\}^8]^2\}^{-1}\end{aligned}$$

For the perimeter of a cell, substituting $\Theta = P$ and

$$p = 2\delta\left(\sin\frac{\alpha_1}{2} + \sin\frac{\alpha_2}{2} + \sin\frac{\alpha_1+\alpha_2}{2}\right)$$

we obtain for the probability density function of the cell perimeter (see Fig. 6)

$$f_P(p)=\frac{\pi}{12}\lambda^2 p^3\int_0^{2\pi}\int_0^{2\pi-\alpha_1}\frac{g(\alpha_1,\alpha_2)}{[\sin(\alpha_1/2)+\sin(\alpha_2/2)+\sin[(\alpha_1+\alpha_2)/2]]^4}$$
$$\times\exp\left(\frac{-\lambda\pi p^2}{4[\sin(\alpha_1/2)+\sin(\alpha_2/2)+\sin[(\alpha_1+\alpha_2)/2]]^2}\right)$$
$$\times d\alpha_2\,d\alpha_1,\qquad p\geqslant 0$$

and (3.3) leads to

$$EP^k=\frac{2^{k+1}\Gamma(2+k/2)}{3\pi^{k/2+1}\lambda^{k/2}}\int_0^{2\pi}\int_0^{2\pi-\alpha_1}g(\alpha_1,\alpha_2)$$
$$\times\left(\sin\frac{\alpha_1}{2}+\sin\frac{\alpha_2}{2}+\sin\frac{\alpha_1+\alpha_2}{2}\right)^k d\alpha_2\,d\alpha_1$$

In particular, the results are

$$EP^1=\frac{32}{3\pi\lambda^{1/2}},\qquad EP^2=\frac{125}{3\pi\lambda}$$

$$EP^3=\frac{225\pi+9216}{20\pi^2\lambda^{3/2}},\qquad EP^4=\frac{13706}{5\pi^2\lambda^2}$$

$$\operatorname{var}P=\frac{375\pi-32^2}{9\pi^2\lambda}$$

$$\operatorname{skw}P=\frac{6075\pi^3-471168\pi+1310720}{20(375\pi-32^2)^{3/2}}$$

$$\operatorname{exc}P=\frac{-3(64800\pi^3+333063\pi^2-5025792\pi+10485760)}{5(375\pi-32^2)^2}$$

An already well-known result is the probability density function $f_L(l)$ of the length of an edge of a Delaunay cell,[11]

$$f_L(l)=\frac{\lambda\pi l}{3}\left(\sqrt{\lambda}\,l\exp\left(\frac{-\lambda\pi l^2}{4}\right)+\frac{2}{\sqrt{\pi}}\right)+\frac{2}{\sqrt{\pi}}\int_{[(\lambda\pi)^{1/2}/2]l}^{\infty}\exp(-x^2)\,dx\Bigg),$$
$$l\geqslant 0$$

Substituting $\Theta = L$ and without loss of generality $l = 2\delta \sin(\alpha_1/2)$, this can be derived from (2.1). The further results are

$$EL^k = \frac{2^{k+1}(k+1)(k+3)\,\Gamma((k+1)/2)}{3(k+2)\,\pi^{(k+1)/2}\lambda^{k/2}}$$

$$\text{var } L = \frac{405\pi - 32^2}{81\pi^2\lambda} \qquad \text{skw } L = \frac{32(10240 - 3159\pi)}{5(405\pi - 32^2)^{3/2}}$$

$$\text{exc } L = \frac{-3(309825\pi^2 - 4313088\pi + 10485760)}{5(405\pi - 32^2)^2}$$

Analogous to that of the angle A in the three-dimensional case, the well-known probability density function $f_A(\alpha)$ of the angle A in a cell spanned by two of its edges can be given, namely

$$f_A(\alpha) = \frac{4}{3\pi} \sin\alpha[\sin\alpha + (\pi - \alpha)\cos\alpha], \qquad 0 \leqslant \alpha < \pi$$

with

$$EA = \frac{\pi}{3}, \qquad EA^2 = \frac{4\pi^2 - 15}{18}$$

$$EA^3 = \frac{\pi^3}{6} - \pi, \qquad EA^4 = \frac{4\pi^4 - 40\pi^2 + 105}{30}$$

$$\text{var } A = \frac{\pi^2}{9} - \frac{5}{6}, \qquad \text{skw } A = \frac{4\pi(\pi^2 - 9)}{(4\pi^2 - 30)^{3/2}}$$

$$\text{exc } A = \frac{3(765 - 8\pi^4)}{5(2\pi^2 - 15)}$$

5. DISCUSSION

The present paper gives an exact analytical description of the behavior of geometrical characteristics of the three-dimensional Delaunay tessellation generated by a stationary Poisson point process. The results are based on a general formula given by Miles;[3] they are obtained by a unified method.

The characteristics investigated here are of great interest in several fields of physics, as evidenced by the fact that many authors have studied them intensively by means of simulation. The special case that the generating point process is Poisson has been summarized in Kumar and Kurtz.[2] The probability density function of $2A$, the double of the angle inside of a face, also has been studied by Lorz[12] and van de Weygaert[13] by simulation.

In contrast to the multiplicity of the simulation studies, analytical results concerning the Delaunay tessellation are given only in a few papers. These are, for example, Miles,[3] with the important formulas describing the size and shape of the Poisson Delaunay cell of an arbitrary dimension, and Møller,[4] who partly obtained Miles' results in another way. Mean values for the three-dimensional Poisson Delaunay cell are given by Okabe *et al.*[1] Analytical expressions for the probability density functions of the angle A in a face are given by Kumar and Kurtz.[2] Rathie[8] has given an analytical expression for the probability density function of the volume V of the three-dimensional Poisson Delaunay cell.

In the present paper, probability density functions have been given for the volume of the three-dimensional Poisson Delaunay cell, for the area and the perimeter of a face, for the edge length, and for an angle inside of a face. These probability density functions are in general threefold integral formulas. The elegant expression given by Rathie[8] is a sequence of complicated analytical standard functions, whereas the integral formulas given in the present paper are more suitable for a numerical evaluation of the probability density functions.

Analytical expressions for higher moments of the equivalent radius R, the area S, and the perimeter P of a face have been given for the first time. A short summary has been given for the one- and two-dimensional cases.

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